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1982 J. Phys. A: Math. Gen. 15 L753

(http://iopscience.iop.org/0305-4470/15/12/017)

J. Phys. A: Math. Gen. 15 (1982) L753–L756. Printed in Great Britain

LETTER TO THE EDITOR

S-matrix for magnons in continuous Heisenberg ferromagnetic chain

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Received 8 September 1982

Abstract. The *S*-matrix for magnons in the continuous Heisenberg chain is obtained with the quantum inverse method. Using the *S*-matrix for the strings of magnons, the position changes and phase shifts of classical solitons in scattering are derived.

In a recent paper we showed that the energy spectrum of magnons of the continuous Heisenberg ferromagnetic chain can be investigated by using the quantum inverse method (Zhao 1982). In this note we will show that, by the same method, the *S*-matrix for the magnons can be obtained, and the scattering of classical solitons can be studied by using the *S*-matrix for the strings of magnons.

The Hamiltonian for the continuous Heisenberg ferromagnetic chain, by choosing suitable units, can be written as

$$H=\int \mathrm{d}x\,(\partial \boldsymbol{s}/\partial x)^2, \tag{1}$$

where $\boldsymbol{s}$ is the spin density and satisfies the commutator

$$[s^j(x), s^k(y)]=\mathrm{i}\varepsilon^{jkl}s^l(x)\,\delta(x-y), \qquad j,k,l=1,2,3. \tag{2}$$

We assume the boundary condition

$$\langle s^3(x)\rangle\to\tfrac{1}{2}, \qquad \langle s^{1,2}(x)\rangle\to 0 \quad \text{as } |x|\to\infty. \tag{3}$$

As in our previous work (Zhao 1982), we first replace the continuous chain by a discrete one with finite length L, and we assume that the lattice spacing Δ is small. We define

$$T(\lambda)=\begin{pmatrix} a(\lambda) & -b^*(\lambda)\\ b(\lambda) & a^*(\lambda)\end{pmatrix}=\lim_{L\to\infty} V(\lambda)^{-N}T_L(\lambda)V(\lambda)^N, \tag{4}$$

where λ is the spectrum parameter, $N=L/\Delta$ is a large integer,

$$V(\lambda)=\begin{pmatrix} 1-\mathrm{i}\lambda\Delta/2 & 0\\ 0 & 1+\mathrm{i}\lambda\Delta/2\end{pmatrix} \tag{5}$$

and

$$T_L(\lambda)=L_N(\lambda)L_{N-1}(\lambda)\dots L_{-N+1}(\lambda). \tag{6}$$

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0305-4470/82/120753+04\$02.00

$L_n(\lambda)$ is a matrix:

$$L_n(\lambda)=\begin{pmatrix} I_n-\mathrm{i}\lambda s_n^3 & -\mathrm{i}\lambda s_n^- \\ -\mathrm{i}\lambda s_n^+ & I_n+\mathrm{i}\lambda s_n^3 \end{pmatrix}, \tag{7}$$

where

$$s_n^i=\int_{x_n-\Delta}^{x_n} \mathrm{d}x\, s^i(x) \quad \text{and} \quad s_n^\pm=s_n^1\pm \mathrm{i}s_n^2.$$

The commutation relations of a, a^*, b and b^* for $\lambda\neq\mu$ have been previously obtained (Zhao 1982). A careful treatment of the limiting process $L\to\infty$ yields the following relations, which are generalisations of those obtained before (Zhao 1982):

$$a(\lambda)b^*(\mu)=\frac{\lambda-\mu-\mathrm{i}\lambda\mu}{\lambda-\mu+\mathrm{i}\varepsilon}\, b^*(\mu)a(\lambda), \tag{8}$$

$$b(\mu)a(\lambda)=\frac{\lambda-\mu-\mathrm{i}\lambda\mu}{\lambda-\mu+\mathrm{i}\varepsilon}a(\lambda)b(\mu), \tag{9}$$

$$b(\lambda)b^*(\mu)=\frac{(\lambda-\mu)^2+(\lambda\mu)^2}{(\lambda-\mu+\mathrm{i}\varepsilon)^2}b^*(\lambda)b(\mu)+2\pi\lambda\mu a^*(\lambda)a(\mu)\,\delta(\lambda-\mu), \tag{10}$$

$$[a(\lambda), a(\mu)]=[a(\lambda), a^*(\mu)]=[b(\lambda), b(\mu)]=0, \tag{11}$$

where ε is infinitesimal and positive. Define $R(\lambda)=a^{-1}(\lambda)b(\lambda)/(\sqrt{2\pi}\lambda)$. From the above relations we have

$$R^*(\lambda)R^*(\mu)=S(\mu,\lambda)R^*(\mu)R^*(\lambda), \tag{12}$$

$$R(\lambda)R^*(\mu)-S(\lambda,\mu)R^*(\mu)R(\lambda)=\delta(\lambda-\mu), \tag{13}$$

where

$$S(\mu,\lambda)=(\mu^{-1}-\lambda^{-1}-\mathrm{i})/(\mu^{-1}-\lambda^{-1}+\mathrm{i}). \tag{14}$$

$R(\lambda)$ and $R^*(\lambda)$ are the annihilation and creation operator, respectively, for a magnon of momentum $p=\tan^{-1}\lambda$. If λ_i is real and $v(\lambda_1)<v(\lambda_2)<\ldots<v(\lambda_n)$, where $v(\lambda_k)$ is the group velocity for the magnon corresponding to the momentum $p_k=\tan^{-1}\lambda_k$, then $|\lambda_1,\lambda_2,\ldots,\lambda_n\rangle=R^*(\lambda_1)\ldots R^*(\lambda_n)|0\rangle$ is an in-state, and $|\lambda_n,\lambda_{n-1},\ldots,\lambda_1|0\rangle$ is an out-state. $R^*(\lambda_1)\ldots R^*(\lambda_n)|0\rangle$ and $b^*(\lambda_1)\ldots b^*(\lambda_n)|0\rangle$ have the same eigenvalue of $a(\lambda)$ but with different normalisation. $S(\mu,\lambda)$ is the S-matrix element for two magnons with $v(\lambda)<v(\mu)$.

From the above discussion, it is easily shown that the magnon number is conserved, and only pure elastic scattering can occur. The S-matrix element for $|\mathrm{in}\rangle=|\lambda_1,\ldots,\lambda_n\rangle$ to $|\mathrm{out}\rangle=|\lambda_n,\ldots,\lambda_1\rangle$ is

$$s(\lambda_n,\ldots,\lambda_1)=\prod_{j<l}\frac{\lambda_l^{-1}-\lambda_j^{-1}-\mathrm{i}}{\lambda_l^{-1}-\lambda_j^{-1}+\mathrm{i}}. \tag{15}$$

The above S-matrix for n particles (magnons) is factorised into two-particle ones. This is a general property of the S-matrix for $1+1$ dimension integrable models which have infinite conservation laws. This property was first found in the δ-function interaction system (Yang 1968).

The magnons can be grouped into strings (bound states). To construct a string we must continue the spectrum parameters into the complex plane. As the magnon number n in a string is large ($n \gg 1$), the string corresponds to a classical soliton (Zhao 1982).

It is easy to write down the S-matrix for the scattering of N strings and of strings–magnons. As an example, we consider the scattering of two strings. The complex spectrum parameters of the two strings are, respectively,

$$\begin{aligned} \lambda_j^{-1} &= \alpha - \tfrac{1}{2}\mathrm{i}j, \qquad & j &= -(m-1), -(m-3), \ldots, (m-1), \\ \mu_l^{-1} &= \beta - \tfrac{1}{2}\mathrm{i}l, \qquad & l &= -(n-1), -(n-3), \ldots, (n-1), \end{aligned} \tag{16}$$

where

$$\alpha = \tfrac{1}{2}[m \cot P + (m^2 \cot^2 P + m^2 - 1)^{1/2}], \qquad \beta = \tfrac{1}{2}[n \cot Q + (n^2 \cot^2 Q + n^2 - 1)^{1/2}]. \tag{17}$$

P and Q are the total momentum of the two strings of magnons respectively, and are real. For $n \gg 1$ and $m \gg 1$, we have $\alpha \approx \frac{1}{2}m \cot \frac{1}{2}P$ and $\beta \approx \frac{1}{2}n \cot \frac{1}{2}Q$. We assume $\partial E(P, m)/\partial P < \partial E(Q, n)/\partial Q$, where E is the energy of a string. The S-matrix element of the scattering of the two strings of magnons is

$$S = \prod_{j,l} \frac{\mu_l^{-1} - \lambda_j^{-1} - \mathrm{i}}{\mu_l^{-1} - \lambda_j^{-1} + \mathrm{i}}. \tag{18}$$

Now, let us consider the classical limit. If $m \gg 1$ and $n \gg 1$, we have approximately

$$\Phi \equiv \mathrm{i} \ln s \approx -2[(\zeta_1^{-1} - \zeta_2^{-1}) \ln(\zeta_1^{-1} - \zeta_2^{-1}) - (\zeta_1^{-1} - \zeta_2^{*-1}) \ln(\zeta_1^{-1} - \zeta_2^{*-1}) + \mathrm{cc}], \tag{19}$$

where $\zeta_1^{-1} = \alpha - \mathrm{i}m/2$, $\zeta_2^{-1} = \beta - \mathrm{i}n/2$. ζ_1 and ζ_2 are the zeros of the eigenvalue of $a(\lambda)$ in the classical limit. Each zero corresponds to a classical soliton.

From classical mechanics we know that the dynamical variables at time $t = -\infty$ can be transformed to those at $t = \infty$ by a canonical transformation. The generating function of this transformation in our case is the classical limit of i*lns*, i.e. Φ. The position changes and the phase shifts of the solitons in scattering can be calculated from Φ (Kulish *et al* 1976). The results are

$$\Delta x_{01} = \frac{-1}{4 \operatorname{Im} \zeta_1} \frac{\partial \Phi}{\partial(\operatorname{Re} \zeta_1^{-1})} = \frac{1}{\operatorname{Im} \zeta_1} \ln \left| \frac{\zeta_1 - \zeta_2}{\zeta_1 - \zeta_2^*} \right|, \tag{20}$$

$$\Delta x_{02} = \frac{-1}{4 \operatorname{Im} \zeta_2} \frac{\partial \Phi}{\partial(\operatorname{Re} \zeta_2^{-1})} = \frac{-1}{\operatorname{Im} \zeta_2} \ln \left| \frac{\zeta_1 - \zeta_2}{\zeta_1 - \zeta_2^*} \right|, \tag{21}$$

$$\Delta \phi_{01} = \frac{1}{2} \frac{\partial \Phi}{\partial(\operatorname{Im} \zeta_2^{-1})} = 2 \arg \frac{\zeta_2^*(\zeta_1 - \zeta_2)}{\zeta_2(\zeta_1 - \zeta_2^*)}, \tag{22}$$

$$\Delta \phi_{02} = \frac{1}{2} \frac{\partial \Phi}{\partial(\operatorname{Im} \zeta_2^{-1})} = -2 \arg \frac{\zeta_1^*(\zeta_1 - \zeta_2)}{\zeta_1(\zeta_1^* - \zeta_2)}, \tag{23}$$

where Δx_{0i} is the position change of the ith soliton, and $\Delta \phi_{0i}$ the phase shift. Equations (20)–(23) coincide with those obtained by Fogedby (1980) in classical theory. (Takhtajan's $\Delta \phi_{0i}$ (Takhtajan 1977) should have a minor correction as pointed out by Fogedby (1980).) Equations (20)–(23) can be easily generalised to the case of N-soliton scattering.

The author would like to thank Professor C N Yang for encouragement and warm hospitality at the Institute for Theoretical Physics, SUNY at Stony Brook, NY. This work is supported in part by the NSF Grant No PHY-81-09110.

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